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QUASI-LINEAR EVOLUTION EQUATIONS IN BANACH SPACES[1]

BY

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ABSTRACT. This paper is concerned with the quasi-linear evolution equation $u'(t) + A(t, u(t))u(t) = 0$ in $[0, T]$, $u(0) = x_0$ in a Banach space setting. The spirit of this inquiry follows that of T. Kato and his fundamental results concerning linear evolution equations. We assume that we have a family of semigroup generators that satisfies continuity and stability conditions. A family of approximate solutions to the quasi-linear problem is constructed that converges to a "limit solution." The limit solution must be the strong solution if one exists. It is enough that a related linear problem has a solution in order that the limit solution be the unique solution of the quasi-linear problem. We show that the limit solution depends on the initial value in a strong way. An application and the existence aspect are also addressed.

This paper is concerned with the quasi-linear evolution equation

$$u'(t) + A(t, u(t))u(t) = 0 \quad \text{in}[0, T], \qquad u(0) = x_0$$

in a Banach space setting.

The spirit of this inquiry follows that of T. Kato. Kato wrote a fundamental paper on linear evolution equations in 1953 **[9]**; that is, investigation of

$$u'(t) + A(t)u(t) = 0 \quad \text{on}[0, T], \qquad u(0) = x_0.$$

He strengthened and extended his analysis of the linear problem in 1970 **[11]**. Kato also wrote on the quasi-linear problem in 1975 **[13]**. We feel that our results give a natural approach to dealing with the quasi-linear problem.

After discussing the setting and method of attack, our theorem is stated and proved. We then give an application of the theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type. A proposition relevant to our theorem is also given.

Let X and Y be Banach spaces, with Y densely and continuously embedded in X. Let $x_0 \in Y, T > 0, r > r_1 > 0$, $r_2 > 0$, $W = \bar{B}_X(x_0; r), Z = B_X(x_0; r_1) \cap B_Y(x_0; r_2)$, and for each $t \in [0, T]$ and $w \in W$, let $-A(t, w)$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators in X, with $Y \subset D(A(t, w))$.

We consider the quasi-linear evolution equation

$$v'(t) + A(t, v(t))v(t) = 0. \tag{QL}$$

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Given a function u from $[0, T']$ into W, where $0 < T' \leqslant T$, we can also consider the linearized evolution equation

$$v'(t) + A(t, u(t))v(t) = 0. \tag{L; u}$$

By a solution (or strong solution) of (QL) or (L; u) on $[0, T']$, we mean a function v on $[0, T']$ to W which is absolutely continuous (X) and differentiable (X) a.e., such that $v(t) \in Y$ a.e., ess $\sup\{\|v(t)\|_Y\} < \infty$, and v satisfies the appropriate equation, (QL) or (L; u), a.e. on $[0, T']$.

Our method is to produce, for each $x_1 \in Z$, a "limit solution" u with initial value x_1 on an interval $[0, T']$, where $T' \in (0, T]$ is independent of x_1. For a partition $\Delta = \{t_0, t_1, \ldots, t_N\}$ of $[0, T']$, we use an iterative procedure to produce a Lipschitz continuous (X) function u_Δ which satisfies

$$u'_\Delta(t) + A(t_i, u_\Delta(t_i))u_\Delta(t) = 0 \quad \text{for } t \in (t_i, t_{i+1})$$

and $i \in \{0, 1, \ldots, N-1\}$, with $u_\Delta(0) = x_1$. This u_Δ is shown to be the time-ordered juxtaposition of the semigroups generated by the $-A(t_i, u_\Delta(t_i))$. These approximate solutions converge uniformly, as $|\Delta|$ goes to 0, to give the limit solution u. We show, in particular, that if $v = w$ is a solution of (QL) or (L; u) on $[0, T']$ with initial value x_1, then $w = u$. Thus, subject to an initial value, a solution of (QL) is unique if it exists, and whenever the linearized equation (L; u) has a solution, then so does the quasi-linear equation (QL). There are known conditions which are sufficient in order that (L; u) has a solution.

Our hypotheses form a natural extension of Kato's assumptions for linear equations **[11]**.

The term "limit solution" seems as appropriate as any to describe the function obtained by the iterative procedure; e.g., see Kobayashi **[16]** or Crandall and Evans **[3]**.

THEOREM. *Assume that*

(i) $\{A(t, w)\}$ *is stable in* X *with constants of stability* M, β; *i.e.*,

$$\|(A(t_k, w_k) + \lambda)^{-1}(A(t_{k-1}, w_{k-1}) + \lambda)^{-1} \ldots (A(t_1, w_1) + \lambda)^{-1}\|_X \leqslant M(\lambda - \beta)^{-k},$$

$\lambda > \beta$, *for any finite family* $\{(t_j, w_j)\}$, $0 \leqslant t_1 \leqslant \ldots \leqslant t_k \leqslant T$, $k = 1, 2, \ldots$.

(ii) $Y \subset D(A(t, w))$ *for each* (t, w), *which implies that* $A(t, w) \in B(Y, X)$, *and the map* $(t, w) \to A(t, w)$ *is Lipschitz continuous with constant* C_1; *i.e.*,

$$\|A(t_2, w_2) - A(t_1, w_1)\|_{Y,X} \leqslant C_1(|t_2 - t_1| + \|w_2 - w_1\|_X).$$

(iii) *There is a family* $\{S(t, w)\}$ *of isomorphisms of* Y *onto* X *such that* $S(t, w) A(t, w) S(t, w)^{-1} = A_1(t, w)$ *is the negative of the infinitesimal generator of a strongly continuous semigroup in* X *for each* (t, w), *and* $\{A_1(t, w)\}$ *is stable in* X, *with constants of stability* M_1, β_1. *Furthermore, there is a constant* C_2 *such that* $\|S(t, w)\|_{Y,X} \leqslant C_2$, $\|S(t, w)^{-1}\|_{X,Y} \leqslant C_2$, *and the map* $(t, w) \to S(t, w)$ *is Lipschitz continuous with constant* C_3 *(see* (ii) *above).*

Then, there exists a T', *with* $0 < T' \leqslant T$, *such that for each* $x_1 \in Z$ *and partition* $\Delta = \{t_0, t_1, \ldots, t_n\}$ *of* $[0, T']$, *we can find a function* u_Δ *which is Lipschitz continuous*

(X) on $[0, T']$ to W, Y-bounded, and satisfies $u'_\Delta(t) + A(t_i, u_\Delta(t_i))u_\Delta(t) = 0$ for $t \in (t_i, t_{i+1})$ and $i \in \{0, 1, \ldots, n-1\}$, with $u_\Delta(0) = x_1$. In fact, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\Delta| < \delta$ implies that $\|u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X < \varepsilon$ except at $t_1, \ldots, t_n$. Further, the u_Δ converge uniformly, as $|\Delta|$ goes to 0, to a Lipschitz continuous (X) function u on $[0, T']$ to W which has initial value x_1 and is bounded, independent of x_1, in the relative completion of Y in X (the set of all points in X that are the limit in X-norm of sequences from Y that are bounded in Y-norm).

If $x_2 \in Z$ and w is constructed as above but with initial value x_2, then $\|u(t) - w(t)\|_X \leq C\|x_1 - x_2\|_X$ for $t \in [0, T']$, with C independent of x_1 and x_2.

Now, if v is a solution of (QL) or (L; u) on $[0, T'']$, where $0 < T'' \leq T'$, with initial value x_1, then $v = u$ on $[0, T'']$, and thus solutions to (QL) or (L; u) are uniquely determined by their initial values.

COROLLARY 1. *If Y is reflexive, then (L; u) has a solution on $[0, T']$ with initial value x_1, and thus u is a solution of (QL) on $[0, T']$ with initial value x_1.*

REMARKS. (1) If $D(A(t, w)) = Y$ for each (t, w) and there is a $\lambda > \beta$ such that $\|(\lambda I + A(t, w))^{-1}\|_{X,Y} \leq C_2$ and $\|\lambda I + A(t, w)\|_{Y,X} \leq C_2$ for each (t, w), then (iii) is satisfied with $S(t, w) = \lambda I + A(t, w)$.

(2) If Y is $A(t, w)$-admissible ($\{\exp(-sA(t, w))\}$ takes Y to Y and forms a strongly continuous semigroup on Y) for each (t, w) and $\{A(t, w)\}$ is stable in Y, then (iii) is unnecessary.

We now begin to prove the Theorem. The proofs of the above remarks and Corollary will be given later.

Let $T^0 = \min(T, r/\|A\|Me^{\beta T}(\|x_0\|_Y + r_2))$, where $\|A\| = \sup\{\|A(t, w)\|_{Y,X}$: $t \in [0, T], w \in W\}$ which is finite by (ii). Let $K = C_2C_3M_1T^0$ and

$$T' = T^0/\left(1 + \|A\|C_2^2M_1e^{K+\beta_1 T}(\|x_0\|_Y + r_2)\right).$$

LEMMA A. *If u is Lipschitz continuous (X) on $[0, T']$ to W with Lipschitz constant*

$$\|A\|C_2^2M_1e^{K+\beta_1 T'}(\|x_0\|_Y + r_2),$$

then $\{A(t, u(t))$: $t \in [0, T']\}$ is Y-stable with constants $C_2^2M_1e^K$ and β_1.

PROOF OF LEMMA A. We use Kato's Proposition 4.4 [11] with $S(t) = S(t, u(t))$. Then we estimate the variation of S by

$$V_S \leq C_3\left(1 + \|A\|C_2^2M_1e^{K+\beta_1 T'}(\|x_0\|_Y + r_2)\right)T' \leq C_3T^0,$$

whence $\{A(t, u(t))\}$ is Y-stable with constants $C_2^2M_1e^{C_2M_1C_3T^0} = C_2^2M_1e^K$ and β_1. This completes the proof of Lemma A.

By an *evolution operator* $\{W(t, s)$: $0 \leq s \leq t \leq T'\}$ generated by $\{\mathcal{A}(t)$: $t \in [0, T']\} \subset \{A(t, w)$: $t \in [0, T'], w \in W\}$ and a partition $\Delta = \{t_0, \ldots, t_N\}$ of $[0, T']$, we mean the family of operators obtained by forming a time-ordered juxtaposition of the semigroups generated at the points of the partition; e.g., for $t \in [t_i, t_{i+1}]$, $s \in [t_j, t_{j+1}]$, $s \leq t$,

$$W(t, s) = \exp(-(t - t_i)\mathcal{A}(t_i))\exp(-(t_i - t_{i-1})\mathcal{A}(t_{i-1})) \ldots \exp(-(t_{j+1} - s)\mathcal{A}(t_j)).$$

It follows from (i) and Kato's Proposition 3.3 **[11]** that $\|W(t, s)\|_X \leqslant Me^{\beta(t-s)}$. If $\{\mathcal{Q}(t)\}$ is Y-stable with constants $\tilde{M}, \tilde{\beta}$, then $W(t, s)Y \subset Y$ and $\|W(t, s)\|_Y \leqslant \tilde{M}e^{\tilde{\beta}(t-s)}$ as a result of (iii) and Kato's Propositions 2.4 and 3.3 **[11]**. Let $\bar{t} = t_i$ if $t \in [t_i, t_{i+1})$, $i \neq N$, and $\bar{t}_N = t_N$. If $f(t) = W(t, 0)x_1$ on $[0, T']$, then f satisfies $f'(t) + \mathcal{Q}(\bar{t})f(t) = 0$ for $t \notin \Delta$, with $f(0) = x_1$. The construction of an evolution operator from a family of semigroup generators and a partition, the notation $\bar{t}$, and the other results above will be used from this point on without further discussion.

LEMMA B. *Suppose $\{\mathcal{Q}(t)$: $t \in [0, T']\}$ is Y-stable with constants $\tilde{M}$ and $\tilde{\beta}$, and that $\{W(t, s)\}$ is generated by $\{\mathcal{Q}(t)\}$ and a partition Δ of $[0, T']$. Then, $f(t) = W(t, 0)x_1$ is Lipschitz continuous (X) with Lipschitz constant $\|A\|\tilde{M}e^{\tilde{\beta}T'}(\|x_0\|_Y + r_2)$.*

The result is also true if $\{W(t, s)\}$ is the evolution operator of Kato's Theorem 4.1 **[11]**.

PROOF OF LEMMA B. For the partition case, since $f'(t) = -\mathcal{Q}(\bar{t})f(t)$ except for $t \in \Delta$, we get for $s \leqslant t$

$$\|f(t) - f(s)\|_X = \left\|-\int_s^t A(\bar{\xi})f(\xi)d\xi\right\|_X \leqslant \|A\|\,\|f\|_Y|t - s|$$
$$\leqslant \|A\|\tilde{M}e^{\tilde{\beta}T'}\|x_1\|_Y|t - s| \leqslant \|A\|\tilde{M}e^{\tilde{\beta}T'}(\|x_0\|_Y + r_2)|t - s|.$$

Now, the f on $[0, T']$ obtained from Kato's evolution operator is the uniform (X) limit of the f corresponding to the partitions Δ as $|\Delta| \to 0$. This establishes the result in the second case and completes the proof of Lemma B.

Together, Lemma A and Lemma B suggest an iteration scheme. We fix x_1 and Δ, then obtain sequences $\{u_n\}$, $\{A_n(t)\}$, and $\{U_n(t, s)\}$, with $A_n(t) = A(t, u_n(t))$, $\{U_{n+1}(t, s)\}$ the evolution operator generated by $\{A_n(t)\}$ and Δ, and $u_{n+1}(t) = U_{n+1}(t, 0)x_1$. Once Lemma A is satisfied, we have $\{A_n(t)$: $t \in [0, T']\}$ is Y-stable with constants $C_2^2M_1e^K$ and β_1; then, Lemma B applied to $\{\mathcal{Q}(t)\} = \{A_n(t)\}$, $\tilde{M} = C_2^2M_1e^K$ and $\tilde{\beta} = \beta_1$, implies that u_{n+1} is Lipschitz continuous (X) on $[0, T']$ with Lipschitz constant $\|A\|C_2^2M_1e^{K+\beta_1T'}(\|x_0\|_Y + r_2)$. Assuming $u_{n+1}[0, T'] \subset W$, the stage is set to apply Lemma A to $\{A_{n+1}(t)$: $t \in [0, T']\}$ and continue the process.

We now work with a fixed partition Δ of $[0, T']$ and fixed $x_1 \in Z$.

Let $A_0(t) = A(t, x_1)$ for $t \in [0, T']$ and let $\{U_1(t, s)\}$ be the evolution operator generated by $\{A_0(t)\}$ and Δ. Define $u_1(t) = U_1(t, 0)x_1$. Then, $u_1'(t) + A_0(\bar{t})u_1(t) = 0$ except at $t_1, t_2, \ldots, t_N$. Also,

$$\|u_1(t) - x_1\|_X = \|U_1(t, t)x_1 - U_1(t, 0)x_1\|_X$$
$$= \left\|\int_0^t U_1(t, s)A_0(\bar{s})x_1\, ds\right\|_X$$
$$\leqslant Me^{\beta T'}\|A\|(\|x_0\|_Y + r_2)t \leqslant r$$

by the choice of T^0 and T'. So, $u_1(t) \in W$ for each $t \in [0, T']$. This argument also works for all the following u_n, $n = 2, 3, \ldots$.

To start the procedure, we apply Lemma A to $u \equiv x_1$ and then Lemma B with $\{\mathcal{Q}(t)\} = \{A_0(t)\}$, $\tilde{M} = C_2^2 M_1 e^K$ and $\tilde{\beta} = \beta_1$, proving that u_1 is Lipschitz continuous (X) on $[0, T']$ with the Lipschitz constant $\|A\| C_2^2 M_1 e^{K+\beta_1 T'}(\|x_0\|_Y + r_2)$.

For the next iteration, let $A_1(t) = A(t, u_1(t))$ for $t \in [0, T']$ and let $\{U_2(t, s)\}$ be the evolution operator generated by $\{A_1(t)\}$ and Δ. Define $u_2(t) = U_2(t, 0)x_1$. Then, $u_2'(t) + A_1(\bar{t})u_2(t) = 0$ except at $t_1, t_2, \ldots, t_N$. As with u_1, $u_2(t) \in W$ for each $t \in [0, T']$.

As we commented before, we can continue in like manner. For convenience of notation, let $M_2 = C_2^2 M_1 e^K$. Then, for $n > 1$, we have

$$\begin{aligned}\|u_{n+1}(t) - u_n(t)\|_X &= \|U_{n+1}(t, 0)x_1 - U_n(t, 0)x_1\|_X \\ &= \left\| -\int_0^t U_{n+1}(t, s)(A_n(\bar{s}) - A_{n-1}(\bar{s}))U_n(s, 0)x_1\, ds \right\|_X \\ &\leqslant Me^{\beta T'}C_1 M_2 e^{\beta_1 T'}(\|x_0\|_Y + r_2)\cdot \int_0^t \|u_n(\bar{s}) - u_{n-1}(\bar{s})\|_X\, ds \\ &\leqslant \left(MM_2 e^{(\beta+\beta_1)T'}C_1(\|x_0\|_Y + r_2)\right)^n \cdot \int_0^t \cdots \int_0^t \|u_1(\bar{s}) - x_0\|_X\, ds \\ &\leqslant \frac{\left(MM_2 e^{(\beta+\beta_1)T'}C_1(\|x_0\|_Y + r_2)T'\right)^n}{n!}\, r.\end{aligned}$$

It follows that there exists a continuous function u_Δ on $[0, T']$ to W such that $u_n \to u_\Delta$ uniformly on $[0, T']$ as $n \to \infty$. The rate of convergence is independent of Δ and x_1.

Now, let $A_\Delta(t) = A(t, u_\Delta(t))$ for $t \in [0, T']$ and let $\{U_\Delta(t, s)\}$ be the evolution operator generated by $\{A_\Delta(t)\}$ and Δ. Define $\hat{u}(t) = U_\Delta(t, 0)x_1$, then $\hat{u}'(t) + A_\Delta(\bar{t})\hat{u}(t) = 0$ except at $t_1, t_2, \ldots, t_N$, and

$$\begin{aligned}\|\hat{u}(t) - u_n(t)\|_X &= \|U_\Delta(t, 0)x_1 - U_n(t, 0)x_1\|_X \\ &= \left\| -\int_0^t U_\Delta(t, s)(A_\Delta(\bar{s}) - A_{n-1}(\bar{s}))U_n(s, 0)x_1\, ds \right\|_X \\ &< Me^{\beta T'}C_1 M_2 e^{\beta_1 T'}(\|x_0\|_Y + r_2)\cdot \int_0^t \|\hat{u}(\bar{s}) - u_{n-1}(\bar{s})\|_X\, ds \\ &< \left(\left(MM_2 e^{(\beta+\beta_1)T'}C_1(\|x_0\|_Y + r_2)T'\right)^n / n!\right)\|\hat{u} - x_0\|_X\end{aligned}$$

which tends to 0 as $n \to \infty$. Thus $\hat{u}(t) = u_\Delta(t)$, $u_\Delta'(t) + A_\Delta(\bar{t})u_\Delta(t) = 0$ except at $t_1, \ldots, t_N$, $u_\Delta(0) = x_1$, u_Δ is Lipschitz continuous (X) with Lipschitz constant $\|A\| C_2^2 M_1 e^{K+\beta_1 T'}(\|x_0\|_Y + r_2)$, and $\|u_\Delta(t)\|_Y = \|U_\Delta(t, 0)x_1\|_Y \leqslant C_2^2 M_1 e^{K+\beta_1 T'}(\|x_0\|_Y + r_2)$, independent of t and x_1.

We now establish that $\{u_\Delta\colon u_\Delta(0) = x_1$ and Δ is a partition of $[0, T']\}$ is a family of approximate solutions to (QL) on $[0, T']$ with initial value x_1. Except for $t_1, \ldots, t_N$, we have

$$\begin{aligned}u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t) &= u_\Delta'(t) + A\left(\bar{t}, u_\Delta(\bar{t})\right)u_\Delta(t) \\ &\quad + \left(A(t, u_\Delta(t)) - A\left(\bar{t}, u_\Delta(\bar{t})\right)\right)u_\Delta(t) \\ &= \left(A(t, u_\Delta(t)) - A\left(\bar{t}, u_\Delta(\bar{t})\right)\right)u_\Delta(t).\end{aligned}$$

So,

$$\begin{aligned}
&\|u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X \\
&\qquad \leqslant C_1\big(|t - \bar{t}| + \|u_\Delta(t) - u_\Delta(\bar{t})\|_X\big)M_2e^{\beta_1 T'}\cdot(\|x_0\|_Y + r_2) \\
&\qquad \leqslant C_1M_2e^{\beta_1 T'}(\|x_0\|_Y + r_2)\cdot\big(1 + M_2e^{\beta_1 T'}\|A\|(\|x_0\|_Y + r_2)\big)|t - \bar{t}| \\
&\qquad = L|t - \bar{t}|.
\end{aligned}$$

where L is independent of t in $[0, T']$ and Δ. Thus $\|u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X \leqslant L|\Delta|$ except at $t_1, t_2, \ldots, t_N$. This verifies that we have a family of approximate solutions.

To show that the $\{u_\Delta\}$ converge as $|\Delta| \to 0$, let Δ_1 and Δ_2 be two partitions of $[0, T']$ with $|\Delta_1|$ and $|\Delta_2|$ small enough that both $\|f'(t) + A(t, f(t))f(t)\|_X < \varepsilon$ and $\|g'(t) + A(t, g(t))g(t)\|_X < \varepsilon$ for $t \in [0, T'] \setminus (\Delta_1 \cup \Delta_2)$, where $f(t) = u_{\Delta_1}(t)$, $g(t) = u_{\Delta_2}(t)$, $f(0) = x_1 = g(0)$, and $\varepsilon > 0$ is fixed. The preceding paragraph allows us to do this. Let $\{V(t, s)\}$ be the evolution operator obtained from Kato's Theorem 4.1 [**11**] for $\{A(t, f(t)) : t \in [0, T']\}$. For $s, t \in [0, T'] \setminus (\Delta_1 \cup \Delta_2)$, $s \leqslant t$, we get

$$\begin{aligned}
g'(s) - f'(s) = {}& (g'(s) + A(s, g(s))g(s)) - (f'(s) + A(s, f(s))f(s)) \\
&- A(s, f(s))(g(s) - f(s)) + (A(s, f(s)) - A(s, g(s)))g(s).
\end{aligned}$$

Moving the third expression on the right to the left side of the equation and applying $V(t, s)$, we get

$$\begin{aligned}
V(t, s)(g'(s) - f'(s)) + V(t, s)A(s, f(s))(g(s) - f(s)) \\
= V(t, s)(g'(s) + A(s, g(s))g(s)) \\
- V(t, s)(f'(s) + A(s, f(s))f(s)) \\
+ V(t, s)(A(s, f(s)) - A(s, g(s)))g(s).
\end{aligned}$$

The left side is simply $\partial V(t, s)(g(s) - f(s))/\partial s$. Integrating both sides in s from 0 to t, evaluating the left side at the endpoints, and recognizing that $V(t, t) = I$, we get

$$\begin{aligned}
&g(t) - f(t) - V(t, 0)(x_1 - x_1) \\
&\qquad = \int_0^t V(t, s)(g'(s) + A(s, g(s))g(s))\, ds \\
&\qquad\quad - \int_0^t V(t, s)(f'(s) + A(s, f(s))f(s))\, ds \\
&\qquad\quad + \int_0^t V(t, s)(A(s, f(s)) - A(s, g(s)))g(s)\, ds.
\end{aligned}$$

So,

$$\begin{aligned}
\|g(t) - f(t)\|_X &\leqslant T'Me^{\beta T'}\varepsilon + T'Me^{\beta T'}\varepsilon + Me^{\beta T'}C_1M_2e^{\beta_1 T'}(\|x_0\|_Y + r_2) \\
&\qquad \cdot \int_0^t \|f(s) - g(s)\|_X\, ds \\
&= L_1\varepsilon + L_2\int_0^t \|g(s) - f(s)\|_X\, ds.
\end{aligned}$$

This implies that

$$\|u_{\Delta_1}(t) - u_{\Delta_2}(t)\|_X = \|g(t) - f(t)\|_X = O(\varepsilon)$$

independent of t in $[0, T']$. Thus, $\{u_\Delta\}$ converges uniformly to a function u on $[0, T']$ to W as $|\Delta| \to 0$. We note that u is Lipschitz continuous (X) with constant $\|A\| C_2^2 M_1 e^{K+\beta_1 T'}(\|x_0\|_Y + r_2)$, $u(0) = x_1$, and u is bounded, independent of x_1, by $C_2^2 M_1 e^{K+\beta_1 T'}(\|x_0\|_Y + r_2)$ in the relative completion of Y in X.

We need to know that u "corresponds" to $\{A(t, u(t)) : t \in [0, T']\}$. Let $\{U(t, s)\}$ be the evolution operator obtained from Kato's Theorem 4.1 **[11]** for $\{A(t, u(t))\}$, and define $\bar{u}(t) = U(t, 0)x_1$. By Lemma A, $\{A(t, u(t))\}$ is Y-stable with constants M_2 and β_1. For any partition Δ of $[0, T']$ we have

$$\begin{aligned}\|\bar{u}(t) - u_\Delta(t)\|_X &= \|U(t, 0)x_1 - U_\Delta(t, 0)x_1\|_X \\ &= \left\| -\int_0^t U(t, s)\big(A(s, u(s)) - A_\Delta(\bar{s})\big)U_\Delta(s, 0)x_1\, ds \right\|_X \\ &\leqslant Me^{\beta T'}C_1\big(|\Delta| + \sup\{\|u(s) - u_\Delta(\bar{s})\| : s \in [0, t]\}\big) \\ &\quad \cdot M_2 e^{\beta_1 T'}(\|x_0\|_Y + r_2)T'.\end{aligned}$$

Since u_Δ converges to u uniformly on $[0, T']$ as $|\Delta| \to 0$, and u_Δ is Lipschitz continuous (X) with a Lipschitz constant that is independent of Δ, we see that $\|\bar{u}(t) - u_\Delta(t)\|_X$ goes to $\|\bar{u}(t) - u(t)\|_X$ and to 0 as $|\Delta| \to 0$. Thus $u(t) = \bar{u}(t) = U(t, 0)x_1$.

Suppose $x_2 \in Z$ and that w_Δ and w are obtained in the same manner as u_Δ and u, except that the initial value for w_Δ and w is x_2. Analogous to the technique employed to obtain u, we get

$$\frac{\partial}{\partial s} U_\Delta(t, s)(u_\Delta(s) - w_\Delta(s)) = U_\Delta(t, s)\big(A(\bar{s}, w_\Delta(\bar{s})) - A(\bar{s}, u_\Delta(\bar{s}))\big)w_\Delta(s)$$

for $s, t \in [0, T']$, $s \leqslant t$, $s \notin \Delta$. Integrating both sides in s from 0 to t yields

$$\begin{aligned}&u_\Delta(t) - w_\Delta(t) - U_\Delta(t, 0)(x_1 - x_2) \\ &\qquad = \int_0^t U_\Delta(t, s)\big(A(\bar{s}, w_\Delta(\bar{s})) - A(\bar{s}, u_\Delta(\bar{s}))\big)w_\Delta(s)\, ds,\end{aligned}$$

and so

$$\begin{aligned}\|u_\Delta(t) - w_\Delta(t)\|_X \leqslant Me^{\beta T'}\|x_1 - x_2\|_X &+ Me^{\beta T'} \cdot C_1 M_2 e^{\beta_1 T'}(\|x_0\|_Y + r_2) \\ &\cdot \int_0^t \|u_\Delta(\bar{s}) - w_\Delta(\bar{s})\|_X\, ds.\end{aligned}$$

Thus, $\|u_\Delta(t) - w_\Delta(t)\|_X \leqslant C\|x_1 - x_2\|_X$, with C independent of t in $[0, T']$, Δ, and x_1 and x_2. It follows that $\|u(t) - w(t)\|_X \leqslant C\|x_1 - x_2\|_X$ for $t \in [0, T']$, with C also independent of the initial values.

We now turn to the uniqueness of solutions to (QL) or (L; u) on $[0, T'']$, where $0 < T'' \leqslant T'$, with initial value $x_1 \in Z$.

Suppose v is such a solution to (L; u). Then, $v'(s) + A(s, u(s))v(s) = 0$ a.e., so

$$\frac{\partial}{\partial s} U(t, s)v(s) = U(t, s)v'(s) + U(t, s)A(s, u(s))v(s) = 0 \quad \text{a.e.}$$

Integrating in s from 0 to t, we get $U(t, t)v(t) - U(t, 0)v(0) = v(t) - U(t, 0)x_1 = v(t) - u(t) = \text{constant}$. Since $v(0) - u(0) = x_1 - x_1 = 0$, we have $v(t) = u(t)$ for all t in $[0, T'']$. This makes u the unique solution of (L; u) on $[0, T'']$ with initial value x_1. In fact, this also makes u a solution of (QL). We note that it is not necessary that $v([0, T'']) \subset W$.

Now suppose that v is a solution to (QL) on $[0, T'']$ with initial value x_1. Then, $v'(s) + A(s, v(s))v(s) = 0$ a.e., and so

$$v'(s) + A(s, u(s))v(s) = (A(s, u(s)) - A(s, v(s)))v(s) \quad \text{a.e.}$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial s} U(t, s)v(s) &= U(t, s)v'(s) + U(t, s)A(s, u(s))v(s) \quad \text{a.e.} \\ &= U(t, s)(A(s, u(s)) - A(s, v(s)))v(s) \quad \text{a.e.} \end{aligned}$$

Integrating in s from 0 to t, we get

$$\begin{aligned} v(t) - u(t) &= U(t, t)v(t) - U(t, 0)v(0) \\ &= \int_0^t U(t, s)(A(s, u(s)) - A(s, v(s)))v(s)\, ds. \end{aligned}$$

This implies that

$$\|v(t) - u(t)\|_X \leq Me^{\beta T'}\|v\|_Y C_1 \int_0^t \|u(s) - v(s)\|_X\, ds,$$

and thus $\|v(t) - u(t)\|_X = 0$ for all t in $[0, T'']$. This makes u the unique solution of (QL) on $[0, T'']$ with initial value x_1. This completes the proof of our Theorem. □

If Y is reflexive, then by Kato's Theorem 5.1 **[11]**, we have the result that $v = u$ is a solution of (L; u) on $[0, T']$ with initial value x_1, and thus u is a solution of (QL). This gives us Corollary 1. □

The remarks following the statement of the Theorem and Corollary are straightforward. We also note that Remark (1) deals with a particular case of condition (iii) of the theorem. Remark (2) contains a condition which greatly simplifies the proof of the Theorem, but which would be extremely difficult to verify in the absence of conditions stronger than condition (iii); e.g., see **[2]**.

We now turn our attention to an application of our Theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type.

COROLLARY 2. *Let S be the sector of the complex plane C consisting of* 0 *and* $\{\lambda \in C: -\theta \leq \arg \lambda \leq \theta\}$, *where $\theta \in (\pi/2, \pi)$ is fixed. We assume that conditions* (i) *and* (ii) *of the Theorem hold with $Y = D(A(t, w))$ for each t, w, and that*

(iii)$'$ *the resolvent set of $-A(t, w)$ contains S and*

$$\|(\lambda I + A(t, w))^{-1}\|_X \leq C_4/(1 + |\lambda|)$$

for each $\lambda \in S$, $t \in [0, T]$, and $w \in W$, where C_4 is a constant independent of λ, t, and w.

Then, the conclusion of the Theorem holds and (L; u) *has a continuously differentiable* (X) *solution on* $[0, T']$ *with initial value x_1, and thus u is a continuously differentiable* (X) *solution of* (QL) *on* $[0, T']$ *with initial value x_1.*

PROOF. Under these conditions the hypotheses of the Theorem hold, where $S(t, w) = A(t, w)$ for each t and w. This gives us the limit solution u. The plan of attack is to produce a solution to (L; u) which is continuously differentiable (X) on

$[0, T']$ and has initial value x_1. This is where the Sobolevskii-Tanabe theory enters. Let $A(t) = A(t, u(t))$ for each $t \in [0, T']$, and we see for $t_1, t_2, t_3 \in [0, T']$ that

$$\begin{aligned}\|(A(t_1) - A(t_2))A(t_3)^{-1}\|_X &\leqslant \|A(t_1) - A(t_2)\|_{Y,X}\|A(t_3)^{-1}\|_{X,Y}\\ &\leqslant C_4\|A(t_1, u(t_1)) - A(t_2, u(t_2))\|_{Y,X}\\ &\leqslant C_5|t_2 - t_1| \quad \text{by (ii)}\end{aligned}$$

and the Lipschitz continuity of u, where C_5 is independent of the choice of t_1, t_2, t_3. It follows from the Sobolevskii-Tanabe theory **[14]**, **[15]**, **[19]**, **[20]**, **[21]**, **[22]** that there is an evolution operator $\{V(t, s): 0 \leqslant s \leqslant t \leqslant T'\}$ such that $v(t) = V(t, 0)x_1$ defines a continuously differentiable (X) function that satisfies $v'(t) + A(t)v(t) = 0$, $v(0) = x_1$. The operator also satisfies $\|A(t)V(t, 0)A(0)^{-1}\|_X \leqslant C_6$ on $[0, T']$, with C_6 independent of t [**19**, p. 5], thus

$$\begin{aligned}\|v(t)\|_Y = \|V(t, 0)x_1\|_Y &= \|A(t)^{-1}A(t)V(t, 0)A(0)^{-1}A(0)x_1\|_Y\\ &\leqslant \|A(t)^{-1}\|_{X,Y}\|A(t)V(t, 0)A(0)^{-1}\|_X \cdot \|A(0)\|_{Y,X}\|x_1\|_Y\\ &\leqslant C_7\|x_1\|_Y,\end{aligned}$$

where C_7 is independent of t. So, except for the image of v lying in W, we have that v is a solution of (L; u). Since the proof of the uniqueness of a solution to (L; u) does not depend on $v([0, T']) \subset W$, we still have that $v(t) = u(t)$ on $[0, T']$. Consequently, u is the solution of (QL) on $[0, T']$ with initial value x_1. In fact, u is continuously differentiable (X), without exception, on $[0, T']$. □

We note that in general an application of the Theorem involves finding conditions that guarantee the existence of a solution to (L; u), which then implies that u is the solution of (QL).

It may be difficult at times to recognize that the conditions for our Theorem hold. The following Proposition gives criteria that obtain the Banach space Y and verify most of condition (iii) of the Theorem. If, in particular, we are able to use $\lambda I + A(t, w)$, where $\lambda > \beta$ is fixed, for $S(t, w)$ in the Proposition, then condition (ii) of the Theorem holds as well as all of condition (iii).

PROPOSITION. *Let Y be a dense linear subspace of X. Suppose for each $t \in [0, T]$ and $w \in W$ that $S(t, w)$ is an isomorphism (algebraically) from Y onto X, $S(t, w)$ is a closed operator in X, $S(t, w)^{-1} \in B(X)$ with $\|S(t, w)^{-1}\|_X \leqslant L_1$, and the bounded linear operator $S(t, w)S(t_0, w_0)^{-1}$ satisfies*

$$\|S(t_2, w_2)S(t_0, w_0)^{-1} - S(t_1, w_1)S(t_0, w_0)^{-1}\|_X \leqslant L_2(|t_2 - t_1| + \|w_2 - w_1\|_X),$$

where L_1, L_2, t_0 from $[0, T]$, and $w_0 \in W$ are fixed. Suppose further that Y has the graph norm induced by $S(t_0, w_0)$; i. e., for $y \in Y$, $\|y\|_Y = \|y\|_X + \|S(t_0, w_0)y\|_X$. Then,

(i) *Y is a Banach space under this norm, and Y is continuously embedded in X.*

(ii) *$S(t, w)^{-1} \in B(X, Y)$ for each t and w, and $\|S(t, w)^{-1}\|_{X,Y} \leqslant 1 + L_1 + L_2(T + 2r)$, where r is the radius of the ball W.*

(iii) *$S(t, w) \in B(Y, X)$ for each t and w, and $\|S(t, w)\|_{Y,X} \leqslant 1 + L_2(T + 2r) \equiv L_3$.*

(iv) *$\|S(t_2, w_2) - S(t_1, w_1)\|_{Y,X} \leqslant L_2L_3(|t_2 - t_1| + \|w_2 - w_1\|_X)$.*

PROOF. Since $S(t_0, w_0)$ is a closed linear operator with domain Y, it is clear that Y is a Banach space under the indicated norm. It is also immediate that Y is continuously embedded in X.

Let $x \in X$, then

$$\begin{aligned}\|S(t, w)^{-1}x\|_Y &= \|S(t, w)^{-1}x\|_X + \|S(t_0, w_0)S(t, w)^{-1}x\|_X \\ &\leq L_1\|x\|_X + \|S(t_0, w_0)S(t, w)^{-1}x - x\|_X + \|x\|_X \\ &\leq (L_1 + 1)\|x\|_X + L_2(|t - t_0| + \|w - w_0\|_X)\|x\|_X \\ &\leq (L_1 + 1)\|x\|_X + L_2(T + 2r)\|x\|_X \\ &= (1 + L_1 + L_2(T + 2r))\|x\|_X.\end{aligned}$$

So, $S(t, w)^{-1} \in B(X, Y)$ and $\|S(t, w)^{-1}\|_{X,Y} \leq 1 + L_1 + L_2(T + 2r)$.

Let $y \in Y$, then

$$\begin{aligned}\|S(t, w)y\|_X &= \|S(t, w)S(t_0, w_0)^{-1}S(t_0, w_0)y\|_X \\ &\leq (1 + L_2(|t - t_0| + \|w - w_0\|_X)) \cdot \|S(t_0, w_0)y\|_X \\ &\leq (1 + L_2(T + 2r))(\|y\|_Y - \|y\|_X) \\ &\leq (1 + L_2(T + 2r))\|y\|_Y.\end{aligned}$$

So, $S(t, w) \in B(Y, X)$ and $\|S(t, w)\|_{Y,X} \leq 1 + L_2(T + 2r)$.

To show (iv), let $y \in Y$, then

$$\begin{aligned}\|S(t_2, w_2)y - S(t_1, w_1)y\|_X &= \|(S(t_2, w_2) - S(t_1, w_1))S(t_0, w_0)^{-1}S(t_0, w_0)y\|_X \\ &\leq \|(S(t_2, w_2) - S(t_1, w_1))S(t_0, w_0)^{-1}\|_X \cdot \|S(t_0, w_0)y\|_X \\ &\leq L_2(|t_2 - t_1| + \|w_2 - w_1\|_X) \cdot \|S(t_0, w_0)\|_{Y,X}\|y\|_Y \\ &\leq L_2L_3(|t_2 - t_1| + \|w_2 - w_1\|_X)\|y\|_Y. \quad \square\end{aligned}$$

We also note that condition (i) for our Theorem holds when each $A(t, w)$ satisfies $\|\exp(-sA(t, w))\|_X \leq e^{\beta t}$.

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